Winter 2017, MATH 215 Calculus III, Exam 2

3/23/2017, 6:10-7:40pm (90 minutes)

- **Your name:** ______________________________

- **Circle your section and write your Lab time:**

Section	Time	Professor	GSI	Lab Time (e.g. Th 10-11)
20	9–10	Sema Gunturkun	Alex Leaf	____________
30	10–11	Mattias Jonsson	Robert Cochrane	____________
40	11–12	Sumedha Ratnayake	Harry Lee	____________
50	12–1	Sumedha Ratnayake	Deshin Finlay	____________
60	1–2	Yueh-Ju Lin	Rebecca Sodervick	____________
70	2-3	Yueh-Ju Lin	Jacob Haley	____________

Instructions:

- This examination booklet contains 7 problems.
- If you want extra space, write on the back..
- **DO NOT remove any sheets or the staple from the exam booklet**.
- **The formula sheet is not collected back and not graded.**
- This is a closed book exam. Electronic devices, calculators, and note-cards are <u>not</u> allowed.
- Show your work and explain clearly.

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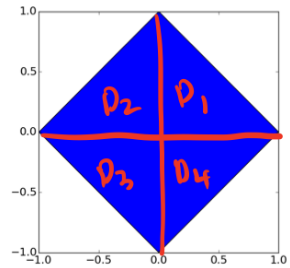
1. (10 points) Find the volume below the surface $z = x^4 + y^4$ and above the square in the x-y plane with vertices at $(x, y) = (\pm 1, 0)$, $(0, \pm 1)$. The square is shown here:

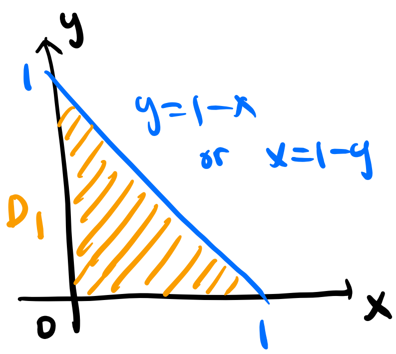


$$\text{Volume} = \iint_D x^4 + y^4 \, dA$$

$x^4 + y^4$ is even in both x and y.

D is symmetric about the x-axis and the y-axis

By symmetry, volume $= 4 \iint_{D_1} x^4 + y^4 \, dA$.



$$D_1 = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le 1-x\}$$
$$= \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1,\ 0 \le x \le 1-y\}$$

$$\iint_{D_1} x^4 + y^4 \, dA = \int_0^1 \int_0^{1-x} x^4 + y^4 \, dy\,dx$$
$$= \int_0^1 x^4 y + \frac{y^5}{5} \Big|_{y=0}^{y=1-x} dx$$
$$= \int_0^1 x^4(1-x) + \frac{(1-x)^5}{5} \, dx$$
$$= \int_0^1 x^4(1-x)\,dx + \int_0^1 \frac{(1-x)^5}{5} \, dx$$

$$\int_0^1 x^4(1-x)\,dx = \int_0^1 x^4 - x^5\,dx = \frac{x^5}{5} - \frac{x^6}{6}\Big|_{x=0}^{x=1} = \frac{1}{30}.$$

$$\int_0^1 \frac{(1-x)^5}{5}\,dx = \int_1^0 -\frac{u^5}{5}\,du = -\frac{u^6}{30}\Big|_{u=1}^{u=0} = \frac{1}{30}.$$

$$\left(\begin{array}{l} u = 1-x \\ du = -dx \end{array}\right)$$

$$\Rightarrow \iint_{D_1} x^4 + y^4\,dA = \frac{1}{30} + \frac{1}{30} = \frac{1}{15}.$$

$$\text{Volume} = 4\iint_{D_1} x^4 + y^4\,dA = \boxed{\frac{4}{15}}$$

<u>Note</u> The domain D remains unchanged upon swapping the x and y axes.

$$\Rightarrow \iint_D x^4\,dA = \iint_D y^4\,dA.$$

$$\text{Volume} = \iint_D x^4 + y^4\,dA = \iint_D x^4\,dA + \iint_D y^4\,dA$$

$$= 2\iint_D x^4\,dA = 8\iint_{D_1} x^4\,dA.$$

This integral is simpler to compute.

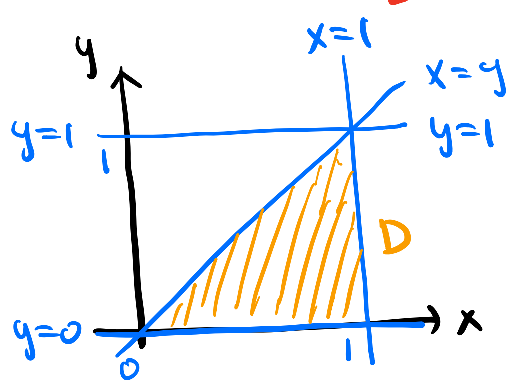
2. Consider the iterated triple integral

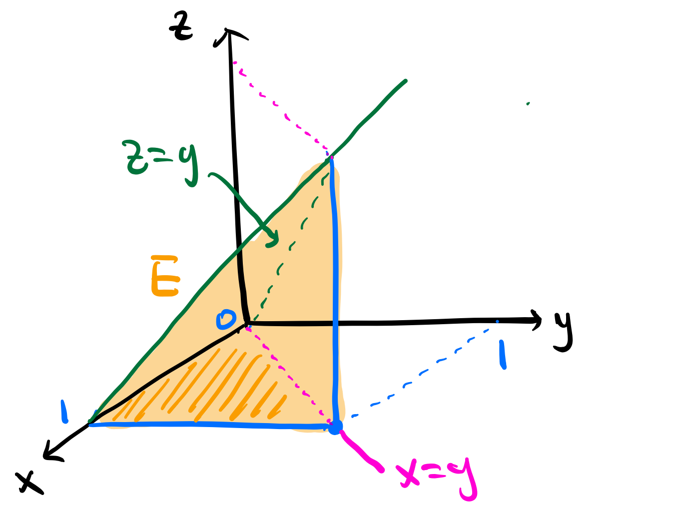
$$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy.$$

In this integral, z is innermost, x is in the middle, and y is outermost.

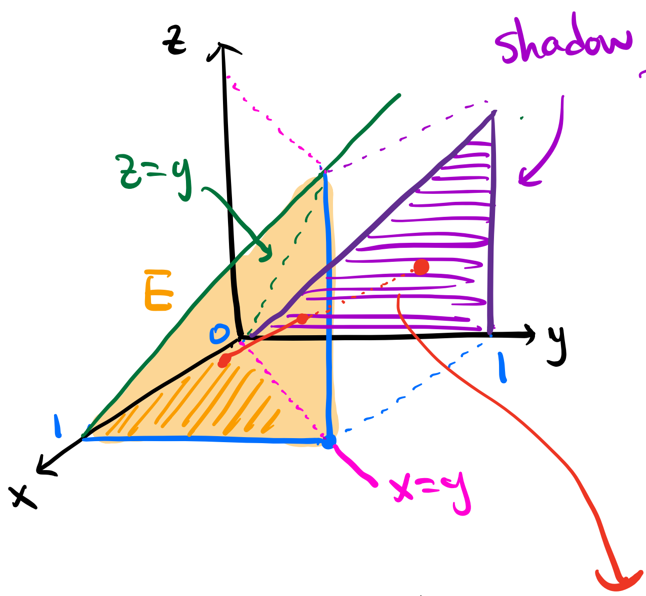
(a) (5 points) Rewrite the integral with x innermost, y in the middle, and z outermost.

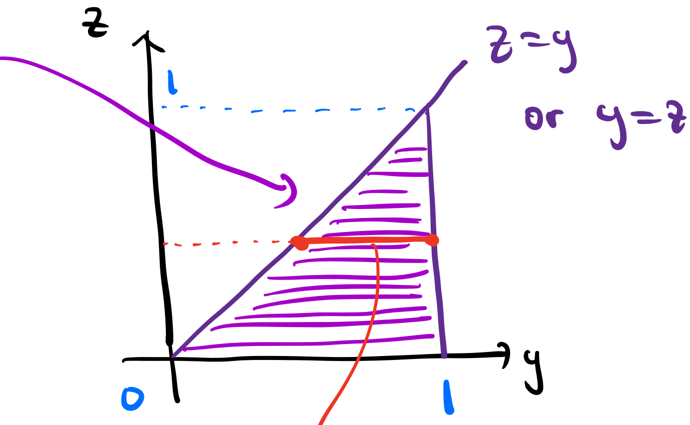
$E = \{(x,y,z) \in \mathbb{R}^3 : \underline{0 \le y \le 1,\ y \le x \le 1},\ 0 \le z \le y\}$





Outer double integral is for $dy\,dz$.





$\rightsquigarrow$ $\underline{0 \le z \le 1,\ z \le y \le 1}$.

For each y and z : $\underline{y \le x \le 1}$

$$\Rightarrow \boxed{\int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz}$$

<u>Note</u> You can also use the cross section method.

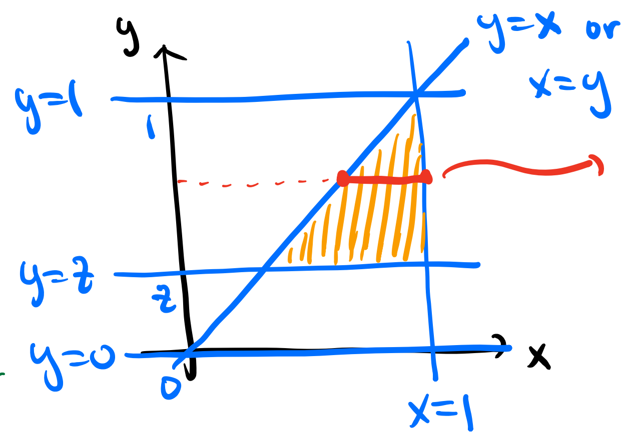
$E = \{(x,y,z) \in \mathbb{R}^3 : 0 \le y \le 1,\ y \le x \le 1,\ 0 \le z \le y\}$

Outermost integral is for dz.

Bounds : $0 \le z \le y \le x \le 1 \Rightarrow \underline{0 \le z \le 1}$.

Inner double integral is for $dxdy$ (z fixed).

Boundary : $0 \le y \le 1,\ y \le x \le 1,\ z \le y$.



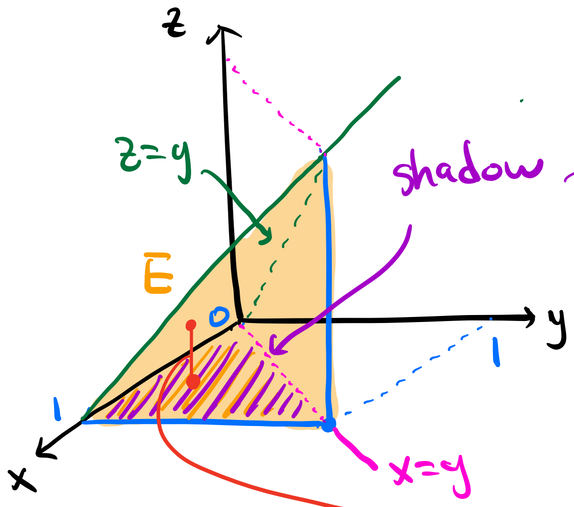
$\underline{z \le y \le 1,\ y \le x \le 1}$.

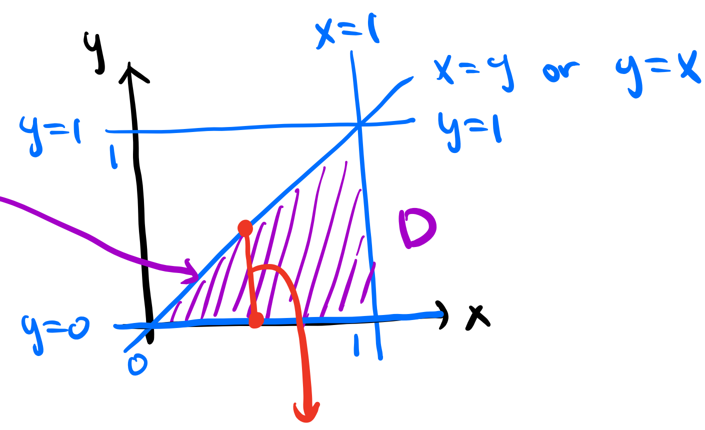
$$\Rightarrow \boxed{\int_0^1 \int_z^1 \int_y^1 f(x,y,z)\, dxdydz}$$

$0 \le z \le 1 \rightsquigarrow$ the line $y=z$ lies between the lines $y=0$ and $y=1$.

(b) (5 points) Rewrite the integral with z innermost, y in the middle, and x outermost.

Outer double integral is for $dy\,dx$.





$0 \le x \le 1,\ 0 \le y \le x$

For each x and y: $0 \le z \le y$.

$$\Rightarrow \boxed{\int_0^1 \int_0^x \int_0^y f(x,y,z)\, dz\, dy\, dx}$$

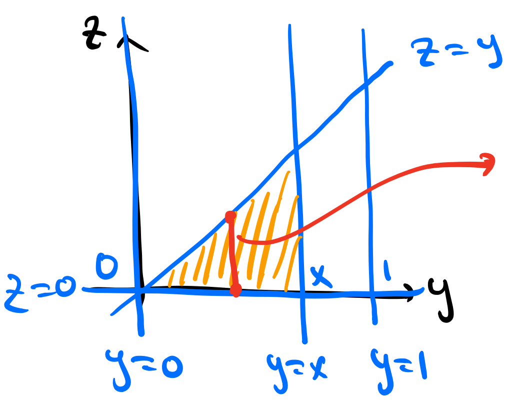
Note You can also use the cross section method.

Outermost integral is for dx

Bounds: $0 \le y \le x \le 1 \Rightarrow 0 \le x \le 1$

Innermost integral is for $dz\,dy$ (x fixed)

Bounds: $0 \le y \le 1,\ y \le x,\ 0 \le z \le y$



$0 \le y \le x,\ 0 \le z \le y$.

3. Let $f(x, y) = x^4 + y^4 + 4xy$.

(a) (5 points) Find three critical points of $f(x, y)$.

$$\nabla f = (f_x, f_y) = (4x^3 + 4y,\ 4y^3 + 4x)$$

$$\nabla f = \vec{0} \Rightarrow \begin{cases} 4x^3 + 4y = 0 \rightsquigarrow \underline{y = -x^3} \\ 4y^3 + 4x = 0 \rightsquigarrow x = -y^3 = x^9 \end{cases}$$

$$x = x^9 \rightsquigarrow x = -1, 0, 1.$$

$$y = -x^3 \Rightarrow (x, y) = \boxed{(-1, 1),\ (0, 0),\ (1, -1)}$$

(b) (5 points) Pick one of the three critical points and classify it as local minimum, local maximum, or saddle.

$$f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x}(4x^3 + 4y) = 12x^2.$$

$$f_{xy} = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}(4x^3 + 4y) = 4$$

$$f_{yy} = \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y}(4y^3 + 4x) = 12y^2$$

$$H = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yx} \end{bmatrix} = \det \begin{bmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{bmatrix}.$$

$$= 144x^2y^2 - 16.$$

At $(-1,1)$: $H = 144 \cdot (-1)^2 \cdot 1^2 - 16 > 0$

$f_{xx} = 12 > 0$.

$\Rightarrow$ $\boxed{\text{A local minimum at } (-1,1)}$

At $(0,0)$: $H = 144 \cdot 0^2 \cdot 0^2 - 16 < 0$

$\Rightarrow$ $\boxed{\text{A saddle point at } (0,0)}$

At $(1,-1)$: $H = 144 \cdot (-1)^2 \cdot 1^2 - 16 > 0$

$f_{xx} = 12 > 0$

$\Rightarrow$ $\boxed{\text{A local minimum at } (1,-1)}$

4. (10 points) Find a critical point (you do not need to classify it as a local maximum or minimum) of

$$f(x, y, z) = -x\log x - 2y\log y - 3z\log z$$

subject to the constraint

$$g(x, y, z) = x + 2y + 3z - 1 = 0.$$

Evaluate f at that point. Here log is the natural logarithm, as usual, so that $\frac{d\log x}{dx} = \frac{1}{x}$.

$$\nabla f = (f_x, f_y, f_z) = (-1-\log x,\ -2-2\log y,\ -3-3\log z)$$

$$\left(f_x = \frac{\partial}{\partial x}(-x\log x) = -1\cdot\log x - x\cdot\frac{1}{x} = -\log x - 1 \right.$$

product rule

Similar computation for f_y and f_z)

$$\nabla g = (g_x, g_y, g_z) = (1, 2, 3)$$

$$\nabla f = \lambda\nabla g : (-1-\log x,\ -2-2\log y,\ -3-3\log z) = \lambda(1,2,3)$$

$$\rightsquigarrow \begin{cases} -1-\log x = \lambda & \Rightarrow \log x = -1-\lambda \Rightarrow x = e^{-1-\lambda} \\ -2-2\log y = 2\lambda & \Rightarrow \log y = -1-\lambda \Rightarrow y = e^{-1-\lambda} \\ -3-3\log z = 3\lambda & \Rightarrow \log z = -1-\lambda \Rightarrow z = e^{-1-\lambda} \end{cases}$$

$$\rightsquigarrow x = y = z.$$

$$g=0 : x + 2y + 3z = 1 \Rightarrow x = y = z = \frac{1}{6}.$$

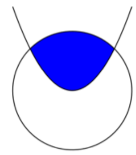
$\Rightarrow$ A critical point is at $\boxed{\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)}$

$$f\left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right) = -\frac{1}{6}\log\left(\frac{1}{6}\right) - \frac{2}{6}\log\left(\frac{1}{6}\right) - \frac{3}{6}\log\left(\frac{1}{6}\right)$$

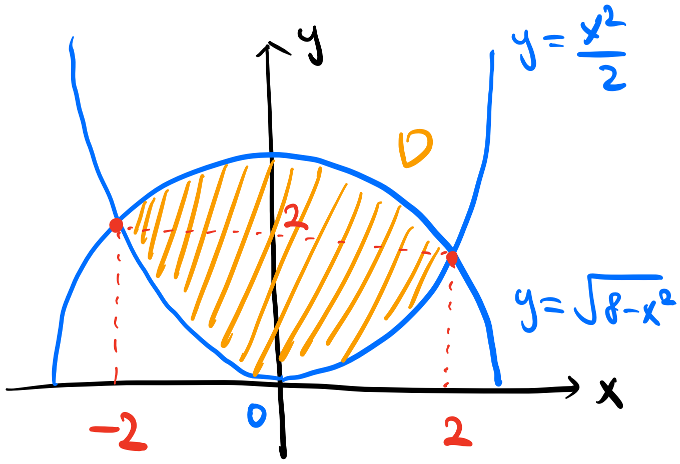
$$= -\log\left(\frac{1}{6}\right) = \boxed{\log(6)}$$

5. (10 points) Find the two points at which the parabola $y = x^2/2$ intersects the circle $x^2 + y^2 = 8$. If D is the region bounded by that parabola and circle (see below), evaluate the double integral:

$$\int\int_D x^2 y\, dx\, dy.$$



The region of integration D looks as follows:
Note: Once you have a numerical answer you *do not* need to simplify it to a fraction. In the textbook, the area element $dx\, dy$ in the integral is given as dA.



Semicircle : $x^2+y^2 = 8$, $y \geq 0 \Rightarrow y = \sqrt{8-x^2}$.

Intersection : $y = \frac{x^2}{2}$ and $x^2+y^2 = 8$

$\Rightarrow x^2 = 2y$ and $x^2 = 8-y^2$

$y \geq 0$

$\rightsquigarrow 2y = 8 - y^2 \rightsquigarrow y = 2, \cancel{-4}$

$\rightsquigarrow x = \pm\sqrt{2y} = \pm 2$.

$\Rightarrow (x,y) = (-2,2), (2,2)$

$D = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 2, \frac{x^2}{2} \leq y \leq \sqrt{8-x^2}\}$.

$$\iint_D x^2y\,dA = \int_{-2}^{2}\int_{x^2/2}^{\sqrt{8-x^2}} x^2y\,dy\,dx$$

$$= \int_{-2}^{2} \frac{x^2y^2}{2}\Bigg|_{y=x^2/2}^{y=\sqrt{8-x^2}}\,dx$$

$$= \int_{-2}^{2} \frac{x^2(8-x^2)}{2} - \frac{x^6}{8}\,dx$$

$$= \int_{-2}^{2} 4x^2 - \frac{x^4}{2} - \frac{x^6}{8}\,dx$$

$$= \frac{4}{3}x^3 - \frac{x^5}{10} - \frac{x^7}{56}\Bigg|_{x=-2}^{x=2}$$

$$= \boxed{\frac{1088}{105}}$$

<u>Note</u> D is symmetric about the y-axis, while the function x^2y is even in x.

$$\Rightarrow \iint_D x^2y\,dA = 2\iint_{D_1} x^2y\,dA$$

where D_1 is the part of D_1 on the first quadrant.

6. Consider the integral

$$\int\int_D \frac{dx\,dy}{(x^2+y^2)^{1/2}},$$

where D is the disc $(x-1)^2+y^2 \le 1$.

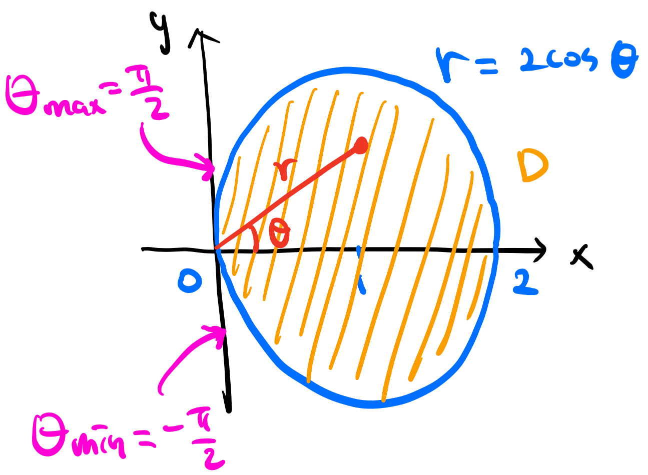
(a) (4 points) Describe the region of integration in polar coordinates.

$(x-1)^2+y^2 \le 1$: a disk of radius 1, centered at $(1,0)$

Write the circle equation in polar coordinates:

$$(x-1)^2+y^2=1 \rightsquigarrow x^2-2x+1+y^2=1 \rightsquigarrow x^2+y^2=2x$$

$$\rightsquigarrow r^2 = 2r\cos\theta \rightsquigarrow r = 2\cos\theta$$



Bounds for θ are given by the y-axis

$\rightsquigarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

For each θ: $0 \le r \le 2\cos\theta$

$$\Rightarrow \boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2\cos\theta}$$

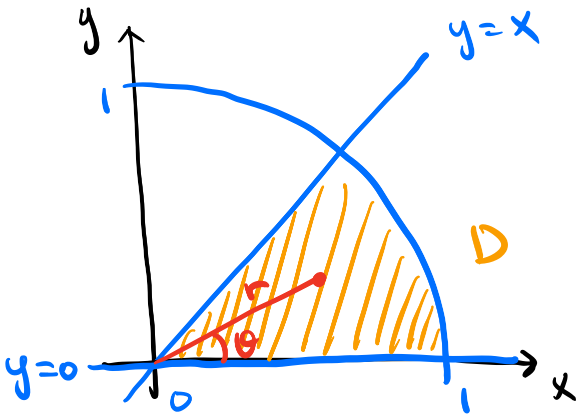
(b) (6 points) Evaluate the integral.

Jacobian

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \frac{1}{\cancel{r}}\cdot\cancel{r}\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2\cos\theta = -2\sin\theta \Big|_{\theta=-\pi/2}^{\theta=\pi/2} = \boxed{4}$$

7. (10 points) Sketch the sector of the unit disc bounded by the lines $x = y$, $y = 0$, and the circle $x^2 + y^2 = 1$ in the first quadrant of the x-y plane. Assuming constant density, find the x and y coordinates of the center of mass.



Bounds for θ are given by the lines $y=0$ and $y=x$

$\theta_{min} = 0$, $\theta_{max} = \tan^{-1}(1) = \frac{\pi}{4}$.

slope

$\Rightarrow 0 \le \theta \le \frac{\pi}{4}$.

For each θ: $0 \le r \le 1$

We may assume that the density $\rho(x,y) = 1$.

$$\text{Mass } m = \iint_D \rho(x,y)\, dA = \iint_D 1\, dA$$

$$= \text{Area}(D) = \frac{\pi}{8}.$$

$$\bar{x} = \frac{1}{m} \iint_D x\,\rho(x,y)\, dA = \frac{1}{\pi/8} \iint_D x\, dA$$

Jacobian

$$= \frac{1}{\pi/8} \int_0^{\pi/4} \int_0^1 r\cos\theta \cdot r\, dr\, d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/4} \int_0^1 r^2 \cos\theta\, dr\, d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/4} \frac{r^3}{3} \cos\theta \Big|_{r=0}^{r=1} d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/4} \frac{1}{3} \cos\theta \, d\theta$$

$$= \frac{8}{3\pi} \sin\theta \Big|_{\theta=0}^{\theta=\pi/4} = \frac{4\sqrt{2}}{3\pi}$$

$$\bar{y} = \frac{1}{m} \iint_D y \rho(x,y) \, dA = \frac{1}{\pi/8} \iint_D y \, dA$$

$$= \frac{8}{\pi} \int_0^{\pi/4} \int_0^1 r\sin\theta \cdot \overset{\text{Jacobian}}{r} \, dr \, d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/4} \int_0^1 r^2 \sin\theta \, dr \, d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/4} \frac{r^3}{3} \sin\theta \Big|_{r=0}^{r=1} d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/4} \frac{1}{3} \sin\theta \, d\theta$$

$$= \frac{8}{3\pi} (-\cos\theta) \Big|_{\theta=0}^{\theta=\pi/4}$$

$$= \frac{8}{3\pi} \left(1 - \frac{\sqrt{2}}{2}\right)$$

Center of mass : $\boxed{\left(\frac{4\sqrt{2}}{3\pi}, \frac{8}{3\pi}\left(1 - \frac{\sqrt{2}}{2}\right) \right)}$